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Ricci Tensor with Six Collineations

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In classifying Ricci tensors in terms of their collineations, an interesting case possessing six collineations arises. These collineations are worked out and discussed.

The classification of spacetime metrics according to their symmetry is important (Petrov, 1969; Ziad, 1990; Bokhari and Qadir, 1987, 1990; Ziad and Qadir, n.d.). The classification of other tensors, e.g., the Riemann tensor, the Ricci tensor, or the Ricci scalar, according to their symmetry may also be important (Kramer *et al.*, 1980). The pioneering work in this direction was done by Katzin *et al.* (1969). More rigorous work was done by Davis *et al.* (1976). In this paper we readdress the same problem in a different perspective. What we intend to do (at some later stage, in the light of the present work) is to classify the general Ricci tensor in terms of its collineations following the line used by Petrov to classify metric tensors. The methods developed by us previously to classify static, spherically symmetric metrics have been employed to work out collineations of the Ricci tensor. It is seen that it gives rise to an interesting case with six collineations. Instead of deriving the full classification here, we restrict ourselves to illustrating the procedure and then deriving the six-collineation case only.

We use the spherically symmetric and static metric tensor (Petrov, 1969) to construct the component of the Ricci tensor and assume that $R_{00}=A(r)$, $R_{11}=B(r)$, $R_{22}=C(r)$, and $R_{33}=C(r)\sin^2\vartheta$.

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The Ricci collineation equation (CE(7)), which we designate by (C_{ab}), is given by

$$(C_{ab}): \quad \xi^c \nabla_c R_{ab} + R_{ac} \nabla_b \xi^c + R_{bc} \nabla_a \xi^c = 0$$

This equation, in a torsion-free space in a coordinate basis, can be further simplified to yield

$$(C_{ab}): \quad \xi^c R_{ab,c} + R_{ac} \xi^c_{,b} + R_{bc} \xi^c{}_{,a} = 0 \tag{1}$$

where $,c$ denotes partial differentiation with respect to the x^c coordinate.

Notice that the (C_{11}) equation can be easily integrated to yield

$$\xi^1 = f(t, \vartheta, \phi)/B^{1/2}, \qquad \text{where} \quad B \neq 0 \tag{2}$$

We use the procedure adopted by us previously (Katzin *et al.*, 1969). There are two possibilities in equation (2) for the values of f, namely (i) $f=0$ and (ii) $f \neq 0$. We restrict ourselves to the second case. Using equation (2) in the CEs (C_{12}) and (C_{22}), differentiating them with respect to ϑ and r, respectively, denoting $D(r) = C'/zCB^{1/2}$ and then comparing gives

$$f(t, \vartheta, \phi)_{\vartheta\vartheta}/f - CD'B^{1/2} = 0 \tag{3}$$

where a prime denotes differentiation with respect to the radial coordinate. Notice that the two terms in the above equation are independent of each other. Thus,

$$f(t, \vartheta, \phi)_{\vartheta\vartheta}/f = CD'B^{1/2} = \alpha \tag{4}$$

where α is a separation constant which may be >0, $=0$, or <0. Another possibility could have been (a) $D'=0$ and (b) $D' \neq 0$. In the first case, it is easy to see (with $\alpha = 0$) that

$$C = \exp\left(a \int B^{1/2}\, dr + b\right) \qquad \text{and} \qquad f = f_1(t, \phi)\vartheta + f_2(t, \phi)$$

where a and b are integration constants. Redefining coordinates, b can be absorbed into the definition and the above equation becomes

$$C = \exp\left(b \int B^{1/2}\, dr\right) \tag{5}$$

Again there are two possibilities in this equation regarding b, i.e., (i) $b=0$ and (ii) $b \neq 0$. In the first case $C=\text{const}$. Since the second case is not of interest at the moment, we will not deal with it here. Now (C_{00}) yields

$$\xi^0{}_{,0} = -(A'/2AB^{1/2})f \tag{6}$$

As above, there are some possibilities here regarding A, i.e., $(*)$ $A'=0$ and $(\dagger)$ $A' \neq 0$. We consider $(*)$ only. Here $A = d$ (a constant) and one can have

(∇) $d>0$, ($\bigcirc$) $d=0$, and ($\Box$) $d<0$. We restrict ourselves to (∇). To derive ($2a*\nabla$), (C_{00}) to (C_{03}) give

$$\xi^0 = F(r, \vartheta, \phi) \tag{7}$$

$$\xi^1 = -dF'\, t/B + F_1(r, \vartheta, \phi) \tag{8}$$

$$\xi^2 = -dF_\vartheta t + F_2(r, \vartheta, \phi) \tag{9}$$

$$\xi^3 = -dF_\phi t/\sin^2\vartheta + F_3(r, \vartheta, \phi) \tag{10}$$

where F, F_1, F_2, and F_3 are integration functions. Substituting equation (12) into (C_{11}) yields

$$d[F'B'/B - 2F'']t + [2B(F_1)' + B'F_1] = 0$$

The above equation is satisfied only if the terms in brackets are separately zero, i.e.,

$$B'/2B - F''/F' = 0 \qquad \text{and} \qquad (F_1)'/F_1 + B'/2B = 0 \tag{11}$$

The above equation can be easily integrated to yield

$$\begin{aligned} F &= K(\vartheta, \phi)\int B^{1/2}\, dr + K_1(\vartheta, \Phi) \\ F_1 &= L(\vartheta, \phi)/B^{1/2} \end{aligned} \tag{12}$$

Substituting equation (13) into (C_{22}) yields

$$\begin{aligned} K &= K_1(\phi)\vartheta + K_{12}(\phi) \\ K_1 &= K_{21}(\phi)\vartheta + K_{22}(\phi) \\ F_2 &= F_4(r, \vartheta) \end{aligned} \tag{13}$$

Incorporating the above into (C_{02}) yields either d or K_{21} equal to zero. Since we are dealing with $d \neq 0$, the obvious choice is that $K_{21}=0$ in equation (16). Similarly, (C_{12}) and (C_{13}) respectively yield

$$L = L_1(\phi) \qquad \text{and} \qquad F_3 = -\left(L_{1\phi}\int B^{1/2}\, dr\right)\Big/\sin^2\vartheta + L_2(\vartheta, \phi)$$

At this stage we substitute all the previously derived results into (C_{23}) to obtain

$$K_{22} = \alpha_1, \qquad K_{11} = \alpha_1, \qquad K_{12} = \alpha_2$$

$$L_1 = \alpha_3, \qquad L = \cot\vartheta\, F_{4\phi} + L_3(\phi)$$

Now for consistency, we use (C_{12}) and (C_{33}). These equations imply that $\alpha_1 = 0$, $L_3 = \alpha_5$, and $F_4 = \alpha_4\cos\phi + \alpha_5\sin\phi$. Relabeling parameters and inserting the above results into the CEs, we obtain

$$\xi^0 = \alpha_0 + \alpha_1 \int B^{1/2}\, dr$$

$$\xi^1 = (\alpha_2 - d\alpha_1\, t)/B^{1/2}$$

$$\xi^2 = (\alpha_3 \cos\phi + \alpha_4 \sin\phi)$$

$$\xi^3 = -\cot\vartheta(\alpha_3 \sin\phi + \alpha_4 \cos\phi) + \alpha_5$$

Notice that the Ricci tensor components in this case are $R_{00} = d = 1$ (5ay), $R_{11} = B(r)$, $R_{22} = 1$, and $R_{33} = \sin^2\vartheta$. Using these components, the Ricci scalar becomes $R = [\bar{e}^{\nu} - \bar{e}^{\lambda}B(r) - 2/r^2]$, which is zero if we choose $B(r) = (e^{\lambda}/\vartheta^2)(r^2 e^{-\nu} - 2)$. From the Einstein field equations with $\Lambda = 0$, the components of the stress energy tensor become

$$T_0^0 = (e^{-\nu} + e^{-\lambda}B)/2 + 1/r^2$$

$$T_1^1 = 1/r^2 - (e^{-\lambda}B + e^{-\nu})/2$$

$$T_2^2 = (e^{-\lambda}B - e^{-\nu})/2 = T_3^3$$

From here it is easily noticed that the spherical symmetry of the metric has not changed this structure of R_{ab}. In fact, it is the same structure as that of $SO(3)$, which arises as a consequence of solving the Killing equations for the spherically symmetric metric. Also present in the above set of equations in ξ^0 is α_0, which corresponds to the time translational invariance due to staticity of the metric. However, there is an extra contribution in the collineation structure due to parameters α_1 and α_2. Thus, it is hoped that if this scheme for the classification of collineations of the general Ricci tensor is considered exhaustively, interesting results can be expected.

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